

On the Stanley depth of powers of some classes of monomial ideals

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Abstract

Given arbitrary monomial ideals I and J in polynomial rings A and B over a field K , we investigate the Stanley depth of powers of the sum $I + J$, and their quotient rings, in $A \otimes_K B$ in terms of those of I and J . Our results can be used to study the asymptotic behavior of the Stanley depth of powers of a monomial ideal. For instance, we solved the case of monomial complete intersection.

Keywords: Stanley depth, monomial ideal, powers of ideals.

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Introduction

Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring over K . Let M be a \mathbb{Z}^n -graded S -module. A *Stanley decomposition* of M is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^r m_i K[Z_i]$ as a \mathbb{Z}^n -graded K -vector space, where $m_i \in M$ is homogeneous with respect to \mathbb{Z}^n -grading, $Z_i \subset \{x_1, \dots, x_n\}$ such that $m_i K[Z_i] = \{um_i : u \in K[Z_i]\} \subset M$ is a free $K[Z_i]$ -submodule of M . We define $\text{sdepth}(\mathcal{D}) = \min_{i=1, \dots, r} |Z_i|$ and $\text{sdepth}_S(M) = \max\{\text{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}$. The number $\text{sdepth}_S(M)$ is called the *Stanley depth* of M . Herzog, Vladoiu and Zheng show in [11] that $\text{sdepth}_S(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. In [18], Rinaldo give a computer implementation for this algorithm, in the computer algebra system CoCoA [7]. For a friendly introduction in the thematic of Stanley depth, we refer the reader [12].

Let I and J be two monomial ideal in polynomial rings A and B over a field K . In [9], H. H. Ha, N. V. Trung and T. N. Trung studied the behaviour of the depth for powers of $I + J$, in terms of those of I and J . Our aim is to do a similar task for the sdepth . However, since there are no homological methods for computing the sdepth , our results are weaker. In Proposition 2.4 we show that $\text{sdepth}((I + J)^n / (I + J)^{n+1}) \geq \min_{i+j=n} \{\text{sdepth}_A(I^i / I^{i+1}) + \text{sdepth}_B(J^j / J^{j+1})\}$, for all $n \geq 0$.

Also, $\text{sdepth}((I + J)^n) \geq \min\{\text{sdepth}(I^n), \text{sdepth}(I^{n-1}J / I^n J), \dots, \text{sdepth}(J^n / I J^n)\}$ for any $n \geq 1$, see Proposition 2.6. There are no general results regarding the asymptotic behavior of the numerical functions $n \mapsto \text{sdepth}_B(B / J^n)$, $n \mapsto \text{sdepth}_B(J^n)$ and $n \mapsto \text{sdepth}_B(J^n / J^{n+1})$, like in the case of depth, see for instance [10]. If $J \subset B$ is a monomial complete intersection ideal, we prove that $\text{sdepth}(R / (I + J)^n) \geq \min_{i \leq n} \{\text{sdepth}_A(A / I^i) + \dim(B / J)\}$, see Theorem 2.11.

In the general case, is difficult to estimate $\text{sdepth}(R / (I + J)^n)$. A partial result for $n = 2$ is given in Proposition 2.9. If $J \subset B$ is a monomial complete intersection, we prove that $\text{sdepth}_B(B / J^n) = \text{sdepth}_B(J^n / J^{n+1}) = \dim(B / J)$, and we give bounds for $\text{sdepth}_B(J^n)$, see Theorem 2.15.

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1 Preliminaries

We consider $I \subset S$ a monomial ideal. We say that I , respectively S/I , satisfies the *Stanley inequality*, if $\text{sdepth}(I) \geq \text{depth}(I)$, respectively $\text{sdepth}(S/I) \geq \text{depth}(S/I)$. Stanley [19] conjectured in fact that for any ideal $I \subset S$, both I and S/I satisfy these inequalities. This conjecture proved to be false in the case of S/I , see [8].

In [16], L. Katthän proposed the following conjecture, which is a weaker form of the Stanley conjecture, and it is open.

Conjecture 1.1. (See [16, Theorem 3.1])

- (1) $\text{sdepth}(S/I) \geq \text{depth}(S/I) - 1$.
- (2) $\text{sdepth}(I) \geq \text{depth}(I)$.

Another conjecture, stated by Herzog in [12], is the following.

Conjecture 1.2. $\text{sdepth}(I) \geq \text{sdepth}(S/I) + 1$.

B. Ichim proved that this conjecture is true for $n \leq 5$, see [14, Theorem 1.2]. Also, this result is true in some other particular cases, for example, when I is a monomial complete intersection, see [3, Theorem 2.4].

Now, we recall the well known Depth Lemma, see for instance [20, Lemma 1.3.9].

Lemma 1.3. (Depth Lemma) *If $0 \rightarrow U \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence of modules over a local ring S , or a Noetherian graded ring with S_0 local, then*

- a) $\text{depth } M \geq \min\{\text{depth } N, \text{depth } U\}$.
- b) $\text{depth } U \geq \min\{\text{depth } M, \text{depth } N + 1\}$.
- c) $\text{depth } N \geq \min\{\text{depth } U - 1, \text{depth } M\}$.

In [17], Asia Rauf proved the analog of Lemma 1.3(a) for sdepth :

Lemma 1.4. *Let $0 \rightarrow U \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of \mathbb{Z}^n -graded S -modules. Then:*

$$\text{sdepth}(M) \geq \min\{\text{sdepth}(U), \text{sdepth}(N)\}.$$

Let $R = K[y_1, \dots, y_m]$ be a polynomial ring. Let M be a graded S -module and N be a graded R -module. It is well known, see for instance [9, Lemma 2.5], that:

$$\text{depth}(M \otimes_K N) = \text{depth}(M) + \text{depth}(N).$$

W. Bruns, C. Krattenthaler, and J. Uliczka proved in [1] a similar result for sdepth :

Lemma 1.5. [1, Proposition 2.10]

$$\text{sdepth}(M \otimes_K N) \geq \text{sdepth}(M) + \text{sdepth}(N)$$

As a consequence of Lemma 1.5, if $\text{sdepth}(M) \geq \text{depth}(M)$ and $\text{sdepth}(N) \geq \text{depth}(N)$, then $\text{sdepth}(M \otimes_K N) \geq \text{depth}(M \otimes_K N)$.

2 Main results

Let $r, s \geq 1$ be two integers, and let $A = K[x_1, \dots, x_r]$ and $B = K[x_{r+1}, \dots, x_{r+s}]$ be polynomial rings over a field K , and let $R = K[x_1, \dots, x_{r+s}]$. Let $I \subset A$ and $J \subset B$ be nonzero proper monomial ideals. In order to simplify the notations, we will denote also by I and J , their extensions in R . We recall the following results of L. T. Hoa and N. D. Ham.

Lemma 2.1. [13, Lemma 1.1 and 3.2] (a) $IJ = I \cap J$.
 (b) $\text{depth}(R/IJ) = \text{depth}_A(A/I) + \text{depth}_B(B/J) + 1$.

As a direct consequence of Lemma 2.1 and [4, Theorem 1.2], we get the following result:

Proposition 2.2. (1) $\text{sdepth}(IJ) \geq \text{sdepth}_A(I) + \text{sdepth}_B(J)$.
 (2) $\text{sdepth}(R/I) \geq \text{sdepth}(R/IJ) \geq \min\{\text{sdepth}(R/I), \text{sdepth}_B(B/J) + \text{sdepth}_A(I)\}$
 (3) $\text{sdepth}(R/J) \geq \text{sdepth}(R/IJ) \geq \min\{\text{sdepth}(R/J), \text{sdepth}_A(A/I) + \text{sdepth}_B(J)\}$
 (4) If I and J satisfy the Stanley inequality, then IJ also satisfy the Stanley inequality.
 (5) If 1.2 holds for I or J , and A/I and B/J satisfy the Stanley inequality, then R/IJ satisfies the Stanley inequality.
 (6) If I and J satisfy 1.1(2) and A/I and B/J satisfy 1.1(1), then R/IJ satisfies 1.1(1).

H. H. Ha, N. V. Trung and T. N. Trung proved in [9] the following result.

Proposition 2.3. $(I + J)^n / (I + J)^{n+1} = \bigoplus_{i+j=n} (I^i / I^{i+1}) \otimes_K (J^j / J^{j+1})$, for all $n \geq 0$.

As a direct consequence of Lemma 1.4, Lemma 1.5 and Proposition 2.3, we get:

Proposition 2.4. For all $n \geq 0$, we have:

$$\text{sdepth}((I + J)^n / (I + J)^{n+1}) \geq \min_{i+j=n} \{\text{sdepth}_A(I^i / I^{i+1}) + \text{sdepth}_B(J^j / J^{j+1})\}. \square$$

For a monomial $v \in S$, we denote $\text{supp}(v) = \{x_j : x_j | v\}$ the support of v .

Proposition 2.5. Let $J \subset B$ be a complete intersection monomial ideal. It holds that $\text{sdepth}(J^n / J^{n+1}) = \dim(B/J)$, for all $n \geq 0$.

Proof. We denote $G(J) = \{v_1, \dots, v_t\}$ the set of minimal monomial generators of J . We use induction on $t \geq 1$. If $t = 1$, then we set $v = v_1$ and we get $J^n / J^{n+1} = (v^n) / (v^{n+1}) = v^n(S/(v))$. Therefore there is nothing to prove. If $t > 1$, we let $B' = K[x_i : i > r, x_i \nmid v_t]$ and $B'' = K[x_i : i > r, x_i | v_t]$. Since J is a complete intersection we get $v_i \in B'$, for $i < t$.

We write $J = J' + J''$ where $J' = (v_1, \dots, v_{t-1}) \subset B'$ and $J'' = (v_t) \subset B''$. Using the induction hypothesis we get $\text{sdepth}(J'^n / J'^{n+1}) = \dim(B'/J')$. By Proposition 2.4, it follows that $\text{sdepth}_B(J^n / J^{n+1}) \geq \dim(B'/J') + \dim(B''/J'') = \dim(B/J)$.

Let $uK[Z] \subset J^n / J^{n+1}$ be a Stanley space. We claim that for each $1 \leq j \leq t$, there exists some $r + 1 \leq i_j \leq r + s$ such that $x_{i_j} \in \text{supp}(v_j) \setminus Z$. Indeed, if $\text{supp}(v_j) \subset Z$, then $uv_j^{n+1} \in uK[Z]$ and $uv_j^{n+1} \in J^{n+1}$, a contradiction. Since the monomials v_j 's have disjoint supports, it follows that $|Z| \leq s - t = \dim(B/J)$ and therefore $\text{sdepth}_B(B/J^n) \leq \dim(B/J)$, as required. \square

Let $Q_i := \sum_{j=0}^i I^{n-j} J^j$, for $0 \leq i \leq n$. Note that there is a chain of ideals:

$$I^n = Q_0 \subset Q_1 \subset \cdots \subset Q_n = (I + J)^n.$$

According to [9, Lemma 2.2], we have $Q_i/Q_{i-1} \cong I^{n-i} J^i / I^{n-i+1} J^i$, for all $1 \leq i \leq n$. Thus, using repeatedly Lemma 1.4, we get the following result.

Proposition 2.6. *For all $n \geq 0$, we have:*

$$\text{sdepth}((I + J)^n) \geq \min\{\text{sdepth}(I^n), \text{sdepth}(I^{n-1}J/I^n J), \dots, \text{sdepth}(J^n/IJ^n)\}. \square$$

Proposition 2.7. *For all $n \geq 0$, we have:*

$$\text{sdepth}((I + J)^n) \geq \min\{\text{sdepth}((I + J)^{n+1}), \text{sdepth}((I + J)^n/(I + J)^{n+1})\}.$$

$$\text{sdepth}(R/(I + J)^{n+1}) \geq \min\{\text{sdepth}(R/(I + J)^n), \text{sdepth}((I + J)^n/(I + J)^{n+1})\}.$$

Proof. We consider the short exact sequences:

$$0 \rightarrow (I + J)^{n+1} \rightarrow (I + J)^n \rightarrow (I + J)^n/(I + J)^{n+1} \rightarrow 0 \text{ and}$$

$$0 \rightarrow (I + J)^n/(I + J)^{n+1} \rightarrow R/(I + J)^{n+1} \rightarrow R/(I + J)^n \rightarrow 0.$$

By Lemma 1.4, we get the required inequalities. \square

Remark 2.8. If we consider the short exact sequence $0 \rightarrow Q_i/Q_{i-1} \rightarrow R/Q_{i-1} \rightarrow R/Q_i \rightarrow 0$, then, by Lemma 1.4, we get $\text{sdepth}(R/Q_{i-1}) \geq \min\{\text{sdepth}(R/Q_i), \text{sdepth}(Q_i/Q_{i-1})\}$, for all $1 \leq i \leq n$.

Also, from the short exact sequence $0 \rightarrow Q_i/Q_{i-1} \rightarrow R/I^{n-i+1}J^i \rightarrow R/I^{n-i}J^i \rightarrow 0$, by Lemma 1.4, we get $\text{sdepth}(R/I^{n-i+1}J^i) \geq \min\{\text{sdepth}(R/I^{n-i}J^i), \text{sdepth}(Q_i/Q_{i-1})\}$, for all $1 \leq i \leq n$.

Proposition 2.9. (1) *If $\text{sdepth}(R/(I^2 + IJ)) < \text{sdepth}_A(A/I) + \text{sdepth}_B(J^2)$, then:*

$$\text{sdepth}(R/(I + J)^2) \leq \text{sdepth}_A(A/I) + \text{sdepth}_B(J^2).$$

(2) *If $\text{sdepth}(R/I^2) < \text{sdepth}_A(I/I^2) + \text{sdepth}_B(J)$, then:*

$$\text{sdepth}(R/(I^2 + IJ)) \leq \text{sdepth}(R/I^2). \text{ Moreover } \text{sdepth}_A(A/I) \leq \text{sdepth}_A(A/I^2).$$

Proof. We consider the following short exact sequences:

$$(i) \quad 0 \longrightarrow J^2/IJ^2 \longrightarrow R/(I^2 + IJ) \longrightarrow R/(I + J)^2 \longrightarrow 0.$$

$$(ii) \quad 0 \longrightarrow IJ/I^2 \longrightarrow R/I^2 \longrightarrow R/(I^2 + IJ) \longrightarrow 0,$$

Note that $IJ/I^2 J \cong (I/I^2) \otimes_K J$ and $J^2/IJ^2 \cong (A/I) \otimes_K J^2$. Therefore, by Lemma 1.5, $\text{sdepth}(IJ/I^2 J) \geq \text{sdepth}_A(I/I^2) + \text{sdepth}_B(J)$ and $\text{sdepth}(J^2/IJ^2) \geq \text{sdepth}_A(A/I) + \text{sdepth}_B(J^2)$.

By (i) and Lemma 1.4, it follows that $\text{sdepth}(R/(I^2 + IJ)) \geq \min\{\text{sdepth}(J^2/IJ^2), \text{sdepth}(R/(I + J)^2)\} \geq \min\{\text{sdepth}_A(A/I) + \text{sdepth}_B(J^2), \text{sdepth}(R/(I + J)^2)\}$, hence we get (1). By (ii) and Lemma 1.4, it follows that

$$\text{sdepth}(R/I^2) = \text{sdepth}_A(A/I^2) + s \geq \min\{\text{sdepth}_A(I/I^2) + \text{sdepth}_B(J), \text{sdepth}(R/(I^2 + IJ))\}.$$

On the other hand, if $\text{sdepth}_A(A/I) > \text{sdepth}_A(A/I^2)$, then, by Lemma 1.4 and the short exact sequence $0 \rightarrow I/I^2 \rightarrow A/I^2 \rightarrow A/I \rightarrow 0$, it follows that $\text{sdepth}_A(I/I^2) \leq \text{sdepth}_A(A/I^2)$, which contradicts the hypothesis of (2). Thus we are done. \square

In the following, we present an example of a monomial ideal $I \subset S$ with $\text{sdepth}_A(A/I) < \text{sdepth}_A(A/I^2)$, firstly given by J. Herzog and T. Hibi in [10], in the frame of depth.

Example 2.10. We consider the ideal $I = (x_1^6, x_1^5x_2, x_1x_2^5, x_2^6, x_1^4x_2^4x_3, x_1^4x_2^4x_4, x_1^4x_2^2x_3^3, x_2^4x_6^2x_3^3)$ in $A = K[x_1, \dots, x_6]$. We have $\text{depth}_A(A/I) = 0$, $\text{depth}_A(A/I^2) = 1$, $\text{depth}_A(A/I^3) = 0$ and $\text{depth}_A(A/I^4) = \text{depth}_A(A/I^5) = 2$.

Using CoCoA [7], we get $\text{sdepth}_A(A/I) = 0$, $\text{sdepth}_A(A/I^2) = 1$ and $\text{sdepth}_A(I) = 3$.

Theorem 2.11. *If $J \subset B$ is a complete intersection monomial ideal, then for all $n \geq 1$:*

- (1) $\text{depth}(R/(I + J)^n) = \min_{i \leq n} \{\text{depth}_A(A/I^i) + \dim(B/J)\}$.
- (2) $r + \dim(B/J) \geq \text{sdepth}(R/(I + J)^n) \geq \min_{i \leq n} \{\text{sdepth}_A(A/I^i) + \dim(B/J)\}$.
- (3) *In particular, if A/I^i satisfy the Stanley inequality (or Conjecture 1(1)), for all $1 \leq i \leq n$, then $R/(I + J)^n$ also satisfies the Stanley inequality (or Conjecture 1(1)).*
- (4) $\text{sdepth}(R/(I + J)^{n+1}) \leq \text{sdepth}(R/(I + J)^n)$, $\text{sdepth}((I + J)^{n+1}) \leq \text{sdepth}((I + J)^n)$ and $\text{sdepth}((I + J)^{n+1}/(I + J)^{n+2}) \leq \text{sdepth}((I + J)^n/(I + J)^{n+1})$, for all $n \geq 1$.
- (5) *The numerical functions $n \mapsto \text{sdepth}(R/(I + J)^n)$, $n \mapsto \text{sdepth}((I + J)^n)$ and $n \mapsto \text{sdepth}((I + J)^n/(I + J)^{n+1})$ are constant for $n \gg 0$.*

Proof. Assume $G(J) = \{v_1, \dots, v_t\}$, where $1 \leq t \leq s$. Note that $\text{depth}_B(B/J^n) = s - t = \dim(B/J)$. We use induction on $t \geq 1$. If $t = 1$, we set $v = v_1$. Then $(I + J)^n = (I, v)^n = I^n + vI^{n-1} + \dots + v^{n-1}I + (v^n)$. We claim that

$$(i) \quad R/(I, v)^n = \bigoplus_{\alpha=0}^{n-1} \bigoplus_{v^\alpha | u, v^{\alpha+1} \nmid u} (A/I^{n-\alpha})u \otimes_K B/(v^\alpha).$$

Indeed, let $w \in R \setminus (I, v)^n$ be a monomial. Let $\alpha := \max\{j \geq 0 : v^j | u\}$ and write $w = v^\alpha u$. Note that $\alpha < n$ and $v \nmid u$. Note that $u \notin I^{n-\alpha}$, otherwise $w \in I^{n-\alpha}v^n \subset (I, v)^n$, a contradiction. Thus, we proved our claim.

Note that $\text{sdepth}_B(B/(v^n)) = \text{depth}_B(B/(v^n)) = s - 1$. By Lemma 1.3 and (i), it follows (1), in the case $t = 1$. Also, by Lemma 1.4, Lemma 1.5 and (i), we get (2), for $t = 1$.

Now, assume $t > 1$. Let $A' = A[x_i : x_i \nmid v_t]$, $B' = K[x_i : x_i | v_t]$ and $B'' = K[x_i : i > r, x_i \nmid v_t]$. Let $J' = (v_t) \subset B'$ and $J'' = (v_1, \dots, v_{t-1}) \subset B''$. We can write $I + J = I' + J'$, where $I' = I + J'' \subset A'$. By induction, we get $\text{depth}_{A'}(A'/I'^n) = \min_{i \leq n} \{\text{depth}_A(A/I^i) + \dim(B''/J'')\}$.

On the other hand, by the first step of induction, we get $\text{depth}(R/(I' + J')^n) = \min_{i \leq n} \{\text{depth}_{A'}(A'/I'^i) + \dim(B'/J')\}$. Since $\dim(B/J) = \dim(B'/J') + \dim(B''/J'')$, we get (1).

Similarly, we get $\text{sdepth}(R/(I + J)^n) \geq \min_{i \leq n} \{\text{sdepth}_A(A/I^i) + \dim(B/J)\}$.

If $uK[Z] \subset R/(I + J)^n$ is a Stanley space, as in the proof of Proposition 2.5, for each $1 \leq j \leq t$ there exists some $r + 1 \leq i_j \leq r + s$ such that $x_{i_j} | v_j$ and $x_{i_j} \notin Z$. Since the monomials v_j 's have disjoint supports, it follows that $|Z| \leq r + s - t$ and therefore $\text{sdepth}(R/(I + J)^n) \leq r + s - t = r + \dim(B/J)$, as required.

(3) Follows immediately from (2).

(4) Using induction on the number of generators of J , it is enough to assume that $J = (u)$ is principal. Since u is regular on R/IR , it follows that $((I, u)^{n+1} : u) = (I, u)^n$. Therefore, by [6, Proposition 2.11], we are done.

(5) Follows immediately from (4). \square

A direct consequence of Theorem 2.11 is the following Corollary.

Corollary 2.12. *If $J \subset B$ is a complete intersection monomial ideal minimally generated by t monomials, then $\text{sdepth}_B(B/J^n) = \text{depth}_B(B/J^n) = s - t$, for all $n \geq 1$. \square*

Corollary 2.13. *Let $L \subset R$ be a monomial ideal with $G(L) = \{v_1, \dots, v_m, v\}$. Assume there exists a monomial $w \in A$ such that $\gcd(v, v_i) = w$ for all $i \leq m$ and $v/w \in B$. Then, $\text{depth}(R/L^n) \leq \text{depth}(R/L^{n+1})$ for all $n \geq 1$.*

Proof. Let $n \geq 1$. Note that $L^n = w^n(L : w)^n$ and therefore $\text{depth}(R/L^n) = \text{depth}(R/(L : w)^n)$. Thus, we can assume that $w = 1$. We write $L = I + J$, where $I = (v_1, \dots, v_m) \subset A$ and $J = (v) \subset B$. We apply Theorem 2.11(1) and we get the required conclusion. \square

A *semilattice* L is a partial ordered set (L, \leq) such that, for any $P, Q \in L$, there is a unique least upper bound $P \vee Q$ called the *join* of P and Q . Let L, L' be two semilattices. A *join-preserving* map $\varphi : L \rightarrow L'$ is a map with $\varphi(P \vee Q) = \varphi(P) \vee \varphi(Q)$.

Let S be a polynomial ring in $n \geq 1$ indeterminates. The *lcm-semilattice* L_G of a finite set $G \subset S$ of monomials is defined as the set of all monomials that can be obtained as the least common multiple (lcm) of some non-empty subset of G , ordered by divisibility. If $I \subset S$ is a monomial ideal, we define the lcm-lattice of I , $L_I := L_{G(I)}$. Note that $u \vee v = \text{lcm}(u, v)$, for any monomials $u, v \in L_I$. See [15] for further details.

Proposition 2.14. *Let $J \subset B$ be a monomial complete intersection ideal with $|G(J)| = t \leq s$. Then $s - t + 1 \leq \text{sdepth}(J^k) \leq s - t + \lceil \frac{t}{k+1} \rceil$, for all $k \geq 1$.*

In particular, if $k \geq t - 1$, then $\text{sdepth}(J^k) = s - t + 1$.

Proof. Let $\mathbf{m} = (x_1, \dots, x_t) \subset S := K[x_1, \dots, x_t]$. Suppose that $J = (v_1, \dots, v_t)$ and let $k \geq 1$. Since v_j 's are monomials with disjoint supports, it follows that L_{J^k} contains all the monomials of the form $v_1^{k_1} \cdots v_t^{k_t}$ with $k_i \geq 0$ and $1 \leq k_1 + \cdots + k_t \leq k$. Also, one can easily see that $\varphi : L_{\mathbf{m}^k} \rightarrow L_{J^k}$, defined by $\varphi(x_1^{k_1} \cdots x_t^{k_t}) := v_1^{k_1} \cdots v_t^{k_t}$, is bijective and join preserving. Thus, according to [15, Theorem 4.5], we have $\text{sdepth}(J^k) = r - t + \text{sdepth}(\mathbf{m}^k)$. On the other hand, according to [5, Theorem 2.2], $1 \leq \text{sdepth}(\mathbf{m}^k) \leq \lceil \frac{t}{k+1} \rceil$. \square

According to [10, Theorem 1.2], if $J \subset B$ is a homogeneous ideal, then there exists a constant $c \geq 0$, such that $\text{depth}_B(B/J^n) = \text{depth}_B(J^n/J^{n+1}) = c$, for $n \gg 0$. One may ask if a similar result is true in the case of sdepth . As a consequence of Proposition 2.5, Corollary 2.12 and Proposition 2.14, we get the following theorem, which give a particular case when the answer is positive.

Theorem 2.15. *If $J \subset B$ is a complete intersection monomial ideal, then:*

- (1) $\text{sdepth}_B(B/J^n) = \text{sdepth}_B(J^n/J^{n+1}) = \dim(B/J)$, for all $n \geq 0$.
- (2) $\text{sdepth}_B(J^n) = \dim(B/J) + 1$, for all $n \geq s - \dim(B/J) - 1$. \square

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